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Korovkin type approximation theorems for weighted $\alpha\beta$-statistical convergence

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Abstract The concept of $\alpha\beta$-statistical convergence was introduced and studied by Aktuğlu (Korovkin type approximation theorems proved via $\alpha\beta$-statistical convergence, J Comput Appl Math 259:174–181, 2014). In this work, we generalize the concept of $\alpha\beta$-statistical convergence and introduce the concept of weighted $\alpha\beta$-statistical convergence of order γ, weighted $\alpha\beta$-summability of order γ, and strongly weighted $\alpha\beta$-summable sequences of order γ. We also establish some inclusion relation, and some related results for these new summability methods. Furthermore, we prove Korovkin type approximation theorems through weighted $\alpha\beta$-statistical convergence and apply the classical Bernstein operator to construct an example in support of our result.

Keywords Korovkin type theorems · Weighted $\alpha\beta$-summability · Rate of the weighted $\alpha\beta$-statistical convergent · Positive linear operator

Mathematics Subject Classification 41A10 · 41A25 · 41A36 · 40A30 · 40G15

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1 Introduction, notations and known results

Let K be a subset of $\mathbb{N}$, the set of natural numbers and $K_n = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n \frac{1}{n}|K_n|$ provided it exists, where $|K_n|$ denotes the cardinality of set K_n. A sequence $x = (x_k)$ is called statistically convergent (st-convergent) to the number ℓ, denoted by $st - \lim x = \ell$, for each $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - \ell| \geq \epsilon\}$ has natural density zero, that is

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \epsilon\}| = 0.$$

The concept of statistical convergence has been defined by Fast [5] and studied by many other authors. It is well know that every statistically convergent sequence is ordinary convergent, but the converse is not true. For example, $y = (y_k)$ is defined as follows;

$$y = (y_k) = \begin{cases} 1, & k = m^2, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that the sequence $y = (y_k)$ is statistical convergent to zero but not convergent. The idea $\alpha\beta$-statistical convergence was introduced by Aktuğlu in [7] as follows:

Let $\alpha(n)$ and $\beta(n)$ be two sequences positive number which satisfy the following conditions

(i) α and β are both non-decreasing,
(ii) $\beta(n) \geq \alpha(n)$,
(iii) $\beta(n) - \alpha(n) \longrightarrow \infty$ as $n \longrightarrow \infty$

and let Λ denote the set of pairs (α, β) satisfying (i)–(iii). For each pair $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$ and $K \subset \mathbb{N}$, we define $\delta^{\alpha,\beta}(K, \gamma)$ in the following way

$$\delta^{\alpha,\beta}(K, \gamma) = \lim_{n\to\infty} \frac{\left|K \cap P_n^{\alpha,\beta}\right|}{(\beta(n) - \alpha(n) + 1)^\gamma},$$

where $P_n^{\alpha,\beta}$ in the closed interval $[\alpha(n), \beta(n)]$. A sequence $x = (x_k)$ is said to be $\alpha\beta$-statistically convergent of order γ to ℓ or $S_{\alpha\beta}^\gamma$-convergent, if

$$\delta^{\alpha,\beta}(\{k : |x_k - \ell| \geq \epsilon\}, \gamma) = \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^\gamma} = 0.$$

In this paper generalizing above idea, we define the weighted $\alpha\beta$-statistical convergence of order γ, the weighted $\alpha\beta$-summability of order γ and the weighted $\alpha\beta$-summability.

2 The weighted $\alpha\beta$-summability

Let $s = (s_k)$ be a sequence of non-negative real numbers such that $s_0 > 0$ and

$$S_n = \sum_{k \in P_n^{\alpha,\beta}} s_k \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty \text{ and } z_n^{\gamma}(x) = \frac{1}{S_n^{\gamma}} \sum_{k \in P_n^{\alpha,\beta}} s_k x_k.$$

Definition 2.1 (a) A sequence $x = (x_k)$ is said to be strongly weighted $\alpha\beta$-summable of order γ to a number ℓ if

$$\lim_{n\to\infty} \frac{1}{S_n^{\gamma}} \sum_{k \in P_n^{\alpha,\beta}} s_k |x_k - \ell| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

We denote it by $x_k \longrightarrow \ell[\overline{N}_{\alpha\beta}^{\gamma}, s]$. Similarly, for $\gamma = 1$ the sequences $x = (x_k)$ is said to be strongly weighted $\alpha\beta$-summable to ℓ. The set of all strongly weighted $\alpha\beta$-summable of order γ and strongly weighted $\alpha\beta$-summable sequences will be denoted $[\overline{N}_{\alpha\beta}^{\gamma}, s]$ and $[\overline{N}_{\alpha\beta}, s]$, respectively.

(b) A sequence $x = (x_k)$ is said to be weighted $\alpha\beta$-summable of order γ to ℓ, if $z_n^{\gamma}(x) \longrightarrow \ell$ as $n \longrightarrow \infty$. Similarly, for $\gamma = 1$ the sequence $x = (x_k)$ is said to be weighted $\alpha\beta$-summable to ℓ, if $z_n(x) \longrightarrow \ell$ as $n \longrightarrow \infty$. The set of all weighted $\alpha\beta$-summable of order γ and weighted $\alpha\beta$-summable sequences will be denoted $(\overline{N}_{\alpha\beta}^{\gamma}, s)$ and $(\overline{N}_{\alpha\beta}, s)$, respectively.

This definition includes the following special cases:

(i) If $\gamma = 1$, $\alpha(n) = 0$ and $\beta(n) = n$, weighted $\alpha\beta$-summable is reduced to weighted mean summable, and $[\overline{N}_{\alpha\beta}, s]$ summable sequences are reduced to $(\overline{N}, p_n)$ summable sequences introduced in [8,14].

(ii) Let λ_n be a none-decreasing sequence of positive numbers tending to ∞ such that $\lambda_n \leq \lambda_n + 1$, $\lambda_1 = 1$. If we take $\gamma = 1$, $\alpha(n) = n - \lambda_n + 1$ and $\beta(n) = n$ then weighted $\alpha\beta$-summability is reduced to $(\overline{N}_{\lambda}; p)$-summability and $[\overline{N}_{\alpha\beta}, s]$ summable sequences are reduced to $[\overline{N}_{\lambda}; p]$-summable sequences introduced in [2].

(iii) If we take $\gamma = 1$, $\alpha(n) = n - \lambda_n + 1$, $\beta(n) = n$ and $s_k = 1$ for all k then $\alpha\beta$-summability is reduced to $(V; \lambda)$-summability introduced in [10] and $[\overline{N}_{\alpha\beta}, s]$-summable sequences are reduced to $[V, \lambda]$-summable sequences introduced in [12].

(iv) Recall that a lacunary sequence $\theta = \{k_r\}$ is an increasing integer sequence such that $k_0 = 0$ and $h_r := k_r - k_{r-1}$. If we take $\gamma = 1$, $\alpha(r) = k_{r-1}+1$ and $\beta(r) = k_r$; then $P^{\alpha,\beta}(r) = [k_{r-1}+1, k_r]$. But because of $[k_{r-1}+1, k_r] \cap \mathbb{N} = (k_{r-1}, k_r] \cap \mathbb{N}$, we have

$$\sum_{k \in [\alpha(n),\beta(n)]} s_k x_k = \sum_{k \in [k_{r-1}+1,k_r]} s_k x_k = \sum_{k \in (k_{r-1},k_r]} s_k x_k.$$

This means that $[\overline{N}_{\alpha\beta}, s]$-summable sequences are reduced to weighted lacunary summable, and $[\overline{N}_{\alpha\beta}, s]$-summable sequence are reduced to strong weighted lacunary summable sequences.

(v) If we take $\gamma = 1$, $\alpha(r) = k_{r-1} + 1$, $\beta(r) = k_r$ and $s_k = 1$ for all k then, weighted $\alpha\beta$-summable is reduced to lacunary summable sequences, and $[\overline{N}_{\alpha\beta}, s]$-summable sequences are reduced to N_θ summbale sequences introduced in [6].

Definition 2.2 A sequence $x = (x_k)$ is said to be weighted $\alpha\beta$-statistically convergent of order γ to ℓ or $S^{\gamma}_{\alpha\beta}$-convergent, if for every $\epsilon > 0$

$$\delta^{\alpha,\beta}(\{k : s_k|x_k - \ell| \geq \epsilon\}, \gamma) = \lim_{n\to\infty} \frac{1}{S_n^{\gamma}} |\{k \leq S_n : s_k|x_k - \ell| \geq \epsilon\}| = 0$$

and denote $st^{\gamma}_{\alpha\beta} - \lim x = \ell$ or $x_k \longrightarrow \ell[\overline{S}^{\gamma}_{\alpha\beta}]$, where $\overline{S}^{\gamma}_{\alpha\beta}$ denotes the set of all weighted $\alpha\beta$-statistically convergent sequences of order γ .

Theorem 2.1 *Let $0 < \gamma \leq \delta \leq 1$. Then, we have $[\overline{N}^{\gamma}_{\alpha\beta}, s] \subseteq [\overline{N}^{\delta}_{\alpha\beta}, s]$ and the inclusion is strict for some γ, δ such that $\gamma < \delta$.*

Proof Let $x = (x_k) \in [\overline{N}^{\gamma}_{\alpha\beta}, s]$ and γ, δ be given such that $0 < \gamma \leq \delta \leq 1$. Then, we obtain that

$$\frac{1}{S_n^{\gamma}} \sum_{k \in P_n^{\alpha,\beta}} s_k|x_k - \ell| \leq \frac{1}{S_n^{\delta}} \sum_{k \in P_n^{\alpha,\beta}} s_k|x_k - \ell|$$

which gives $[\overline{N}^{\gamma}_{\alpha\beta}, s] \subseteq [\overline{N}^{\delta}_{\alpha\beta}, s]$. Now, we show that this inclusion is strict. Let us consider the sequence $t = (t_k)$ defined by,

$$t = (t_k) = \begin{cases} 1, & \beta(n) - \sqrt{\beta(n) - \alpha(n) + 1} + 1 \leq k \leq \beta(n), \\ 0, & \text{otherwise.} \end{cases}$$

If we choose $s_k = 1$ for all $k \in \mathbb{N}$, it is clear that

$$\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}} \sum_{k \in P_n^{\alpha,\beta}} |t_k - 0| \leq \frac{\sqrt{\beta(n) - \alpha(n) + 1}}{(\beta(n)-\alpha(n) + 1)^{\gamma}} = \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma-1/2}}.$$

Since $\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma-1/2}} \longrightarrow 0$ as $n \longrightarrow \infty$ for $1/2 < \beta \leq 1$, we have $t = (t_k) \in [\overline{N}^{\gamma}_{\alpha\beta}, s]$. On the other hand, we get

$$\frac{\sqrt{\beta(n) - \alpha(n) + 1} - 1}{(\beta(n) - \alpha(n) + 1)^{\delta}} \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^{\delta}} \sum_{k \in P_n^{\alpha,\beta}} |t_k - 0|$$

and $\frac{\sqrt{\beta(n)-\alpha(n)+1}-1}{(\beta(n)-\alpha(n)+1)^{\delta}} \longrightarrow \infty$ as $n \longrightarrow \infty$ for $0 < \delta < 1/2$ then, we have $t = (t_k) \notin [\overline{N}^{\delta}_{\alpha\beta}, s]$. This completes the proof. □

Theorem 2.2 *Let $(\alpha, \beta) \in \Lambda$. Then, we have following statements:*

(a) If a sequence $x = (x_k)$ is strongly weighted $(\alpha\beta)$-summable of order γ to limit ℓ, then it is weighted $\alpha\beta$-statistically convergent of order γ to ℓ, that is $[\overline{N}^{\gamma}_{\alpha\beta}, s] \subseteq \overline{S}^{\gamma}_{\alpha\beta}$ and this inclusion is strict.
(b) If $x = (x_k)$ bounded and weighted $\alpha\beta$-statistically convergent of order γ to ℓ then $x_k \longrightarrow [\overline{N}^{\gamma}_{\alpha\beta}, s]$.

Proof (a) Let $\epsilon > 0$ and $x_k \longrightarrow \ell[\overline{N}^{\gamma}_{\alpha\beta}, s]$. Then, we get

$$\frac{1}{S_n^{\gamma}} \sum_{k \in P_n^{\alpha,\beta}} |x_k - \ell| = \frac{1}{S_n^{\gamma}} \sum_{\substack{k \in P_n^{\alpha,\beta} \\ s_k|x_k - \ell| \geq \epsilon}} s_k|x_k - \ell| + \frac{1}{S_n^{\gamma}} \sum_{\substack{k \in P_n^{\alpha,\beta} \\ s_k|x_k - \ell| < \epsilon}} s_k|x_k - \ell| \geq \frac{\epsilon \left| K_{s_n}^{\alpha\beta}(\epsilon) \right|}{S_n^{\gamma}}.$$

This implies that $\lim_{n \to \infty} \frac{\left| K_{s_n}^{\alpha\beta}(\epsilon) \right|}{S_n^{\gamma}} = 0$ which means $\delta^{\alpha,\beta}(K_{s_n}^{\alpha\beta}(\epsilon), \gamma) = 0$, where $K_{s_n}^{\alpha\beta}(\epsilon) = \{k \leq S_n : s_k|x_k - \ell| \geq \epsilon\}$. Therefore, $x = (x_k)$ is weighted $\alpha\beta$-statistically convergent of order γ to ℓ. To prove $[\overline{N}^{\gamma}_{\alpha\beta}, s] \subseteq S^{\gamma}_{\alpha\beta}$ in (a) is strict, let the sequence $x = (x_k)$ be defined by

$$x_k = \begin{cases} k, & 1 \leq k \leq [(\beta(n) - \alpha(n) + 1)^{\gamma/2}], \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Then x is not bounded and for every $\epsilon > 0$. Let $s_k = 1$ for all k. Then we have

$$\begin{aligned} &\frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \leq \beta(n) - \alpha(n) + 1 : |x_k - 0| \geq \epsilon\}| \\ &= \frac{[(\beta(n) - \alpha(n) + 1)^{\gamma/2}]}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

That is, $x_k \longrightarrow 0[S^{\gamma}_{\alpha\beta}]$. But

$$\begin{aligned} &\frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \sum_{k \in P_n^{\alpha,\beta}} |x_k - 0| \\ &= \frac{[(\beta(n) - \alpha(n) + 1)^{\gamma/2}] \left([(\beta(n) - \alpha(n) + 1)^{\gamma/2}] + 1\right)}{2(\beta(n) - \alpha(n) + 1)^{\gamma}} \longrightarrow \frac{1}{2}, \end{aligned}$$

i.e., $x_k \nrightarrow 0[\overline{N}^{\gamma}_{\alpha\beta}, s]$.

(b) Assume that $x = (x_k)$ is bounded and weighted $\alpha\beta$-statistically convergent of order γ to ℓ. Then for $\epsilon > 0$, we have $\delta^{\alpha,\beta}(K_{\epsilon}^{\alpha\beta}, \gamma) = 0$. Since $x = (x_k)$ is bounded, there exists $M > 0$ such that $s_k|x_k - \ell| \leq M$ for all $k \in \mathbb{N}$.

$$\frac{1}{S_n^\gamma}\sum_{k\in P_n^{\alpha,\beta}} s_k|x_k-\ell| = \frac{1}{S_n^\gamma}\sum_{\substack{k\in P_n^{\alpha,\beta}\\ s_k|x_k-\ell|\geq\varepsilon}} s_k|x_k-\ell| + \frac{1}{S_n^\gamma}\sum_{\substack{k\in P_n^{\alpha,\beta}\\ s_k|x_k-\ell|<\varepsilon}} |x_k-\ell|$$
$$\leq \frac{M}{S_n^\gamma}\left|\left\{k\in P_n^{\alpha,\beta} : s_k|x_k-\ell|\geq\epsilon\right\}\right| + \varepsilon,$$

this implies that $x_k \longrightarrow [\overline{N}_{\alpha\beta}^\gamma, s]$.

□

3 Application to Korovkin type approximation

In this section, we get an analogue of classical Korovkin Theorem by using the concept of $\alpha\beta$-statistical convergence. Also we estimate, in rates of $\alpha\beta$-statistical convergence. Recently, such types of approximation theorems are proved, [1,3,4,11,13,14].

Let $C[a,b]$ be the linear space of all real-valued continuous functions f on $[a,b]$ and let L be a linear operator which maps $C[a,b]$ into itself. We say L is positive operator, if for every non-negative $f\in C[a,b]$, we have $L(f,x)\geq 0$ for $x\in[a,b]$. It is well-known that $C[a,b]$ is a Banach space with the norm given by

$$\| f \|_{C[a,b]} = \sup_{x\in[a,b]} |f(x)|.$$

The classical Korovkin approximation theorem states as follows (see [7,9])

$$\lim_{n\to\infty} \| L_n(f,x) - f(x) \|_{C[a,b]} = 0 \Leftrightarrow \lim_{n\to\infty} \| L_n(f,x) - e_i) \|_{C[a,b]} = 0,$$

where $e_i = x^i$ and $f\in C[a,b]$.

Theorem 3.1 *Let (L_k) be a sequence of positive linear operator from $C[a,b]$ in to $C[a,b]$. Then for all $f\in C[a,b]$*

$$\overline{S}_{\alpha\beta}^\gamma - \lim_{k\to\infty} \| L_k(f,x) - f(x)) \|_{C[a,b]} = 0 \tag{3.1}$$

if and only if

$$\overline{S}_{\alpha\beta}^\gamma - \lim_{k\to\infty} \| L_k(e_0,x) - e_0 \|_{C[a,b]} = 0, \tag{3.2}$$
$$\overline{S}_{\alpha\beta}^\gamma - \lim_{k\to\infty} \| L_k(e_1,x) - e_1) \|_{C[a,b]} = 0, \tag{3.3}$$
$$\overline{S}_{\alpha\beta}^\gamma - \lim_{k\to\infty} \| L_k(e_2,x) - e_2) \|_{C[a,b]} = 0. \tag{3.4}$$

Proof Because of $e_i\in C[a,b]$ for $(i=0,1,2)$, conditions (3.2)–(3.4) follow immediately from (3.1). Let the conditions (3.2)–(3.4) hold and $f\in C[a,b]$. By the continuity of f at x, it follows that for given $\varepsilon>0$ there exists δ such that for all t

$$|f(x)-f(t)|<\varepsilon, \quad \text{whenever} \quad \forall|t-x|<\delta. \tag{3.5}$$

Since f is bounded, we get

$$|f(x)| \leq M, \quad -\infty < x, t < \infty.$$

Hence

$$|f(x) - f(t)| \leq 2M, \quad -\infty < x, t < \infty. \tag{3.6}$$

By using (3.5) and (3.6), we have

$$|f(x) - f(t)| < \varepsilon + \frac{2M}{\delta^2}(t-x)^2, \ \forall |t-x| < \delta.$$

This implies that

$$-\varepsilon - \frac{2M}{\delta^2}(t-x)^2 < f(x) - f(t) < \varepsilon + \frac{2M}{\delta^2}(t-x)^2.$$

By using the positivity and linearity of (L_k), we get

$$L_k(1,x)\left(-\varepsilon - \frac{2M}{\delta^2}(t-x)^2\right) < L_k(1,x)\,(f(x)-f(t)) < L_k(1,x)\left(\varepsilon + \frac{2M}{\delta^2}(t-x)^2\right)$$

where x is fixed and so $f(x)$ is constant number. Therefore,

$$\begin{aligned} -\varepsilon L_k(1,x) - \frac{2M}{\delta^2} L_k((t-x)^2, x) &< L_k(f,x) - f(x)L_k(1,x) \\ &< \epsilon L_k(1,x) + \frac{2M}{\delta^2} L_k((t-x)^2, x). \end{aligned} \tag{3.7}$$

On the other hand

$$\begin{aligned} L_k(f,x) - f(x) &= L_k(f,x) - f(x)L_k(1,x) + f(x)L_k(1,x) - f(x) \\ &= [L_k(f,x) - f(x)L_k(1,x) - f(x)L_k] + f(x)[L_k(1,x) - 1]. \end{aligned} \tag{3.8}$$

By inequality (3.7) and (3.8), we obtain

$$L_k(f,x) - f(x) < \varepsilon L_k(1,x) + \frac{2M}{\delta^2} L_k((t-x)^2, x) + f(x) + f(x)[L_k(1,x) - 1]. \tag{3.9}$$

Now, we compute second moment

$$\begin{aligned} L_k((t-x)^2, x) &= L_k(x^2 - 2xt + t^2, x) \\ &= x^2 L_k(1,x) - 2x L_k(t,x) + L_k(t^2, x) \\ &= [L_k(t^2,x) - x^2] - 2x[L_k(t,x) - x] + x^2[L_k(1,x) - 1]. \end{aligned}$$

By (3.9), we have

$$\begin{aligned}
L_k(f,x)-f(x) &< \varepsilon L_k(1,x)+\frac{2M}{\delta^2}\{[L_k(t^2,x)-x^2]-2x[L_k(t,x)-x]\\
&\quad +x^2[L_k(1,x)-1]\}+f(x)(L_k(1,x)-1)\\
&= \varepsilon[L_n(1,x)-1]+\varepsilon+\frac{2M}{\delta^2}\{[L_k(t^2,x)-x^2]-2x[L_k(t,x)-x]\\
&\quad +x^2[L_k(1,x)-1]\}+f(x)(L_k(1,x)-1).
\end{aligned}$$

Because of ε is arbitrary, we obtain

$$\begin{aligned}
\| L_k(f,x)-f(x)\|_{C[a,b]} &\leq \left(\varepsilon+M+\frac{2Mb^2}{\delta^2}\right)\| L_k(e_0,x)-e_0\|_{C[a,b]}\\
&\quad +\frac{4Mb}{\delta^2}\| L_k(e_1,x)-e_1\|_{C[a,b]}+\frac{2M}{\delta^2}\| L_k(e_2,x)-e_2\|_{C[a,b]}\\
&\leq R\left(\| L_k(e_0,x)-e_0\|_{C[a,b]}+\| L_k(e_1,x)-e_1\|_{C[a,b]}\right.\\
&\quad \left.+\| L_k(e_2,x)-e_2\|_{C[a,b]}\right)
\end{aligned}$$

where $R=\max\left(\varepsilon+M+\frac{2Mb^2}{\delta^2},\frac{4Mb}{\delta^2}\right)$.

For $\varepsilon'>0$, we can write

$$\begin{aligned}
\mathcal{C} &:= \left\{k\in\mathbb{N}:\| L_k(e_0,x)-e_0\|_{C[a,b]}+\| L_k(e_1,x)-e_1\|_{C[a,b]}+\| L_k(e_2,x)-e_2\|_{C[a,b]}\geq\frac{\varepsilon'}{R}\right\},\\
\mathcal{C}_1 &:= \left\{k\in\mathbb{N}:\| L_k(e_0,x)-e_0\|_{C[a,b]}\geq\frac{\varepsilon'}{3R}\right\},\\
\mathcal{C}_2 &:= \left\{k\in\mathbb{N}:\| L_k(e_1,x)-e_1\|_{C[a,b]}\geq\frac{\varepsilon'}{3R}\right\},\\
\mathcal{C}_3 &:= \left\{k\in\mathbb{N}:\| L_k(e_2,x)-e_2\|_{C[a,b]}\geq\frac{\varepsilon'}{3R}\right\}.
\end{aligned}$$

Then, $\mathcal{C}\subset\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$, so we have $\delta^{\alpha,\beta}(\mathcal{C},\gamma)\leq\delta^{\alpha,\beta}(\mathcal{C}_1,\gamma)+\delta^{\alpha,\beta}(\mathcal{C}_2,\gamma)+\delta^{\alpha,\beta}(\mathcal{C}_3,\gamma)$. Thus, by conditions (3.2)–(3.4), we obtain

$$\overline{S}^{\gamma}_{\alpha\beta}-\lim_{k\to\infty}\| L_k(f,x)-f(x)\|_{C[a,b]}=0.$$

which completes the proof. □

We remark that our Theorem 3.1 is stronger than that of classical Korovkin approximation theorem. For this claim, we consider the following example:

Example 1 Considering the sequence of Bernstein operators

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{n}{k}\right)\binom{n}{k}x^k(1-x)^{n-k};\ x\in[0,1].$$

We define the sequence of linear operators as $T_n : C[0,1] \longrightarrow C[0,1]$ with $T_n(f,x) = (1+x_n)B_n(f,x)$, where $x = (x_n)$ is defined in (2.1). Now let $s_k = 1$ for all k. Note that the sequence $x = (x_n)$ is weighted $\alpha\beta$-statistically convergent but not convergent. Then, $B_n(1,x) = 1$, $B_n(t,x) = x$ and $B_n(t^2,x) = x^2 + \frac{x-x^2}{n}$ and sequence (T_n) satisfies the conditions (3.2)–(3.4). Therefore, we get

$$\overline{S}^{\gamma}_{\alpha\beta} - \lim_{n\to\infty} \| T_n(f,x) - f(x) \|_{C[a,b]} = 0.$$

On the other hand, we have $T_n(f,0) = (1+x_n)f(0)$, since $B_n(f,0) = f(0)$, thus we obtain

$$\| T_n(f,x) - f(x) \|_{\infty} \geq |T_n(f,0) - f(0)| \geq x_n|f(0)|.$$

One can see that (T_n) does not satisfy the classical Korovkin theorem, since $x = (x_n)$ is not convergent.

4 Rate of weighted $\alpha\beta$-statistically convergent of order γ

In this section, we estimate the rate of weighted $\alpha\beta$-statistically convergent of order γ of a sequence of positive linear operator which is defined from $C[a,b]$ into $C[a,b]$.

Definition 4.1 Let (u_n) be a positive non-increasing sequence. We say that the sequence $x = (x_k)$ is $\alpha\beta$-statistically convergent of order γ to ℓ with the rate $o(u_n)$ if for every, $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{u_n S_n^{\gamma}} |\{k \leq S_n : s_k|x_k - \ell| \geq \epsilon\}| = 0.$$

At this stage, we can write $x_k - \ell = \overline{S}^{\gamma}_{\alpha\beta} - o(u_n)$.

Before proceeding further, let us give basic definition and notation on the concept of the modulus of continuity.

The modulus of continuity of f, $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) = \sup_{\substack{|x-y|\leq\delta \\ x,y\in[a,b]}} |f(x) - f(y)|.$$

It is well-known that for a function $f \in C[a,b]$,

$$\lim_{n\to 0^+} \omega(f,\delta) = 0$$

for any $\delta > 0$

$$|f(x) - f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{4.1}$$

Theorem 4.1 *Let* (L_k) *be sequence of positive linear operator from* $C[a,b]$ *into* $C[a,b]$*. Assume that*

$$
\begin{aligned}
&(i) \quad \| L_k(1,x) - x \|_{C[a,b]} = \overline{S}^{\gamma}_{\alpha\beta} - o(u_n),\\
&(ii) \quad \omega(f,\psi_k) = \overline{S}^{\gamma}_{\alpha\beta} - o(v_n) \text{ where } \psi_k = \sqrt{L_k[(t-x)^2, x]}.
\end{aligned}
$$

Then for all $f \in C[a,b]$*, we get*

$$
\| L_k(f,x) - f(x) \|_{C[a,b]} = \overline{S}^{\gamma}_{\alpha\beta} - o(z_n)
$$

where $z_n = \max\{u_n, v_n\}$.

Proof Let $f \in C[a,b]$ and $x \in [a,b]$. From (3.8) and (4.1), we can write

$$
\begin{aligned}
|L_k(f,x) - f(x)| &\le L_k(|f(t) - f(x)|; x) + |f(x)||L_k(1,x) - 1|\\
&\le L_k\left(\frac{|x-y|}{\delta} + 1; x\right)\omega(f,\delta) + |f(x)||L_k(1,x) - 1|\\
&\le L_k\left(\frac{(t-x)^2}{\delta^2} + 1; x\right)\omega(f,\delta) + |f(x)||L_k(1,x) - 1|\\
&\le \left(L_k(1,x) + \frac{1}{\delta^2}L_k\left((t-x)^2; x\right)\right)\omega(f,\delta) + |f(x)||L_k(1,x) - 1|\\
&= L_k(1,x)\omega(f,\delta) + \frac{1}{\delta^2}L_k\left((t-x)^2; x\right)\omega(f,\delta) + |f(x)||L_k(1,x) - 1|.
\end{aligned}
$$

By choosing $\sqrt{\psi_k} = \delta$, we get

$$
\begin{aligned}
&\| L_k(f,x) - f(x) \|_{C[a,b]} \le \| f \|_{C[a,b]} \| L_k(1,x) - x \|_{C[a,b]}\\
&\quad + 2\omega(f,\psi_k) + \omega(f,\psi_k) \| L_k(1,x) - x \|_{C[a,b]}\\
&\le H\{\| L_k(1,x) - x \|_{C[a,b]} + \omega(f,\psi_k) + \omega(f,\psi_k) \| L_k(1,x) - x \|_{C[a,b]}\},
\end{aligned}
$$

where $H = \max\{2, \| f \|_{C[a,b]}\}$. By Definition 4.1 and conditions (i)–(ii), we get the desired result. □

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